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A CASE OF ITERATION IN SEVERAL VARIABLES.

By Albert A. Bennett.

The Relation between Difference Equations and Iteration.

1. If $F_1(x)$ be a given function, the equation

$$f(x+1) = F_1[f(x)] \tag{1}$$

is a functional equation for $f(x)$, of the form of a difference equation, of the first order. The general difference equation of the first order is of the form

$$f(x+1) = H[f(x), x],$$

where H is a given function of two variables. The difference equation (1) has in any solution $f(x)$ obviously one arbitrary constant at least, since we may choose for the value of $f(x_0)$, say, y_0; the values of $f(x_0 + n)$, where n is any positive (or negative) integer, will then be determinate. There remain, however, an infinite number of parameters in $f(x)$, since the values of $f(x_0 + n)$ for n any real number between zero and one are still arbitrary. If we have any solution $f(x)$, we shall have

$$f(x+2) = F_1\{F_1[f(x)]\} \equiv F_2[f(x)]$$

and if we define $F_n(x)$ for n, a positive integer, by the relation

$$F_n(x) \equiv F_{n-1}[F_1(x)] \equiv F_1[F_{n-1}(x)],$$

we shall then have for n a positive integer,

$$f(x+n) = F_n[f(x)]. \tag{2}$$

We may use (2) for the purpose of defining $F_n(x)$ for n other than a positive integer, if once $f(x+n)$ be defined for all real values of n, and the same feature of arbitrary definition for $0 < n < 1$ appears in $F_n(x)$.

Conversely, starting with $F_1(x)$, we may determine $F_n(x)$ for n a positive or negative integer and define it arbitrarily for every value of n between zero and one. The real solutions of

$$f(x+n) = F_n[f(x)],$$

if $F_1(x)$ be a real function defined for all real values of x, will contain but a single arbitrary constant. The problem of defining or determining $F_n(x)$ from $F_1(x)$ is said to be a problem of iteration. Thus, for the case of a single variable, the study of iteration is essentially the study of a special type of difference equation of the first order.

We shall find that in the case of several variables, by a mere increase in the number of variables, we may make any system of difference equations each of finite order, equivalent to a problem in iteration. Let us consider a set of p difference equations of orders $q_1, q_2, \cdots, q_p$, respectively. Let one of these equations be of the form

$$f(x+q) = H[f(x+q-1), f(x+q-2), \cdots, f(x+1), f(x); x].$$

By a change of notation, we may put

$$\begin{aligned} x &\equiv f_{(0)}(x), \\ f(x) &\equiv f_{(1)}(x), \\ f(x+1) &\equiv f_{(2)}(x), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot \\ f(x+q-1) &\equiv f_{(q)}(x), \end{aligned}$$

so that we have the system

$$\begin{aligned} f_{(0)}(x+1) &= 1 + f_0(x), \\ f_{(1)}(x+1) &= f_2(x), \\ f_{(2)}(x+1) &= f_3(x), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_{(q-1)}(x+1) &= f_q(x), \\ f_{(q)}(x+1) &= H[f_{(q)}(x), f_{(q-1)}(x), \cdots, f_2(x), f_1(x); f_0(x)]. \end{aligned}$$

In this system, we note that x itself no longer appears explicitly. By repeating this process for each of the p equations, and by using throughout a single notation for x, viz., $f_{(0)}(x)$, we obtain a system of $1 + q_1 + q_2 + \cdots + q_p$ difference equations in the same number of unknown functions f, and each of the first order. Thus far, we have implied by our notation, that the number of variables occurring in any of the functions f is one, and that these variables consist simply of x itself. If there be k variables, $x_1, x_2, \cdots, x_k$, appearing in the unknown functions f, and in the functions H, we shall proceed as above and write

$$x_i \equiv f_{(0,\,i)}(x_1, x_2, \cdots, x_k), \qquad i = 1, 2, \cdots, k,$$

and so obtain a system of $k + q_1 + q_2 + \cdots + q_p = m$ difference equations of the first order, in which the x's occur only within the functions f. After this normalization, there will be m equations in m unknown functions, each equation being of the first order, while the unknown functions will contain k variables, $x_1, x_2, \cdots x_k$. We may next suppose that $k = m$. For if $k < m$, we may introduce as additional parameters into our unknown functions, $m - k$ new variables, $x_{k+1}, x_{k+2}, \cdots, x_m$; while if $m < k$, we introduce new equations of the first order in new unknown functions of the same k variables, and so obtain a system for which the number of functions and the number of equations are equal to the number of variables. The reduction may be performed with respect to one variable at a time. Any system of difference equations of arbitrary orders in which the dependent variables for the greatest values of the arguments appear explicitly, together with certain new parameters, and functions, which may be taken in some simple form, may be reduced to a system of the following type:

$$
\begin{aligned}
&f_{(1)}(x_1 + 1, x_2 + 1, \cdots, x_m + 1) \\
&\qquad \equiv F_{(1)1}[f_{(1)}(x_1, \cdots, x_m), f_{(2)}(x_1, \cdots, x_m), \cdots, f_{(m)}(x_1, \cdots, x_m)], \\
&f_{(2)}(x_1 + 1, x_2 + 1, \cdots, x_m + 1) \\
&\qquad \equiv F_{(2)1}[f_{(1)}(x_1, \cdots, x_m), f_{(2)}(x_1, \cdots, x_m), \cdots, f_{(m)}(x_1, \cdots, x_m)], \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&f_{(m)}(x_1 + 1, x_2 + 1, \cdots, x_m + 1) \\
&\qquad \equiv F_{(m)1}[f_{(1)}(x_1, \cdots, x_m), f_{(2)}(x_1, \cdots, x_m), \cdots, f_{(m)}(x_1, \cdots, x_m)],
\end{aligned}
\tag{3}
$$

the identity being in terms of the variables $x_1, x_2, \cdots, x_m$. The equations (3) we shall write symbolically as

$$f(x + 1) = F_1[f(x)]. \tag{4}$$

As in the case of one variable, we may regard the equation, now symbolic,

$$f(x + n) = F_n[f(x)],$$

either as defining F_n in terms of $f(x + n)$, or $f(x + n)$ in terms of F_n, the first being a problem of differences, and the second, of iteration.

A Certain Class of Series and their Iteration.

2. We shall investigate equations (3) in the case in which $F_{(i)1}[u_1, u_2, \cdots, u_m]$ is of the form

$$F_{(i)1}[u_1, u_2, \cdots, u_m] \equiv au_i + ((u^2)), \qquad i = 1, 2, \cdots, m, \tag{5}$$

where by $((u^k))$ is meant a power series in the m variables $(u_1, u_2, \cdots, u_m)$, for which no term appears of order less than k in the m variables taken together. In (5), the constant a is supposed to be the same for each value of $i = 1, 2, \cdots, m$, and for the present we shall suppose that $|a| \neq 0, \neq 1$.

Let us suppose that we are given an arbitrary constant γ, such that $|\gamma| \neq 0$, and let us seek to determine the coefficients of m series, C, where $C_{(i)}$ is of the form

$$C_{(i)}[u_1, u_2, \cdots, u_m] \equiv \gamma u_i + ((u^2)), \quad i = 1, 2, \cdots, m,$$

such that the C's satisfy the symbolic equation

$$C[F_1(u)] = F_1[C(u)].$$

The coefficients of C will be completely determined in terms of γ, and of the coefficients of F_1. For let us form a product of h u's distinct or different and call this product U. Then U will occur once and only once in each $F_{(i)1}$ with coefficients, respectively, say, d_i, and once and only once in each $C_{(i)}$ with coefficients, say, c_i. The coefficient of the term in U in $C_{(i)}[F_1(u)]$ will be of the form

$$\gamma d_i + c_i a^h + (\textit{terms containing neither } d_i \textit{ nor } c_i).$$

The coefficient of the term in U in $F_{(i)1}[C(u)]$ will be of the form

$$a c_i + d_i \gamma^h + (\textit{terms containing neither } d_i \textit{ nor } c_i).$$

Furthermore c_i occurs for the first time in either $C[F_1(u)]$ or $F[C(u)]$ with the term in U, while all of the other terms in the coefficient of U, except that one containing d_i will contain exclusively coefficients of C and F, which appeared in these respective sets of series with terms in u's of degree less than h. Since we are supposing that the coefficients of F are known, and that, for the induction, the coefficients in C of terms in u's of degree less than h, are already determined, we have, to determine c_i, an equation of the form

$$c_i(a^h - a) = d_i(\gamma^h - \gamma) + (\textit{known terms}).$$

Since $|a| \neq 0, \neq 1$, we see that c_i, also, is completely determined. The above argument does not apply to the initial term, and it is readily verified that γ is indeed arbitrary provided that $|\gamma| \neq 0$.

Now $F_n(u)$ must be, itself, a special case of $C(u)$, since

$$F_n[F_1(u)] = F_1[F_n(u)] = F_{n+1}(u).$$

For n an integer, the initial terms of $F_n(x)$ are such that

$$F_{(i)n} = a^n u_i + ((u^2)), \qquad i = 1, 2, \cdots, m. \tag{6}$$

We may choose for n other than an integer, say,

$$F_{(i)n} = a^{n+P(n)} u_i + ((u^2)),$$

where $P(n)$ is any periodic function of n of period unity and vanishing for $n = 0$. For the sake of simplicity, we shall choose for our definition of $F_{(i)n}$, a series of the form (6) for all values of n, a choice which will be seen to be possible. In other words we shall choose for $F_n(u)$ the series $C(u)$ for which $\gamma = a^n$. We shall need to use the fact that the form of $F_{(i)n}(u)$ is completely determined by this choice, and that the coefficient of any term in $F_{(i)n}(u)$ is a polynomial in a^n, while n enters in no other manner into the series.

Application of Newton's Interpolation Formula.

3. We shall now apply Newton's interpolation formula to the expansion of $F_{(i)n}(u)$, and we shall prove that the series obtained from this formula, in a certain manner, about to be described, is not only, itself, a power series, but is identically equal to the form which would be obtained by the method of undetermined coefficients, as just considered.

Newton's interpolation formula, for $y = f(x)$, when the values $y_i = f(x_i)$ are known for $i = 0, 1, 2, 3, \cdots$, is the following:

$$
\begin{aligned}
f(x) &= y_0 + (x - x_0)[x_0x_1] + (x - x_0)(x - x_1)[x_0x_1x_2] \\
&\qquad + (x - x_0)(x - x_1)(x - x_2)[x_0x_1x_2x_3] + \cdots,
\end{aligned} \tag{7}
$$

where

$$
[x_0x_1] = \frac{y_1 - y_0}{x_1 - x_0}, \qquad [x_1x_2] = \frac{y_2 - y_1}{x_2 - x_1}, \qquad [x_2x_3] = \frac{y_3 - y_2}{x_3 - x_2},
$$

$$
[x_0x_1x_2] = \frac{[x_1x_2] - [x_0x_1]}{x_2 - x_0}, \qquad [x_1x_2x_3] = \frac{[x_2x_3] - [x_1x_2]}{x_3 - x_1},
$$

$$
[x_0x_1x_2x_3] = \frac{[x_1x_2x_3] - [x_0x_1x_2]}{x_3 - x_0}
$$

The only fact that we shall use concerning this formula is that it reduces identically to y_n if we put $x = x_n$. To apply (7) to the question in hand, we shall put $y = f(x) \equiv F_{(i)n}(u)/a^n$, $x = a^n$, for every value of n, while the values of x_j, $j = 0, 1, \cdots$, will be, respectively, $a^0, a^1, \cdots, a^j, \cdots$. The formula then becomes,

$$
\begin{aligned}
\frac{F_{(i)n}}{a^n} = \Bigg\{ &\frac{F_{(i)0}}{1} + \frac{(a^n - 1)}{(a - 1)}\left(\frac{F_{(i)1}}{a} - \frac{F_{(i)0}}{1}\right) \\
&+ \frac{(a^n - 1)(a^n - a)}{(a - 1)(a^2 - 1)}\left(\frac{F_{(i)2}}{a^3} - \frac{F_{(i)1}}{a^2} - \frac{F_{(i)1}}{a} + \frac{F_{(i)0}}{1}\right) + \cdots \Bigg\}
\end{aligned} \tag{8}
$$

or as it may be written,

$$
(9)\qquad \begin{aligned} F_{(i)n} = a^n\Bigg[V_0 + \frac{a^n - 1}{a - 1} V_1 + \frac{(a^n - 1)(a^n - a)}{(a - 1)(a^2 - 1)} V_2 + \cdots \\ + \frac{(a^n - 1)\cdots(a^n - a^{j-1})}{(a - 1)\cdots(a^j - 1)} V_j + \cdots \Bigg], \end{aligned}
$$

where $V_0 \equiv F_{(i)0}$ *and*

$$
V_j(u_1, u_2, \cdots, u_m) \equiv \frac{V_{j-1}[F_{(1)1}, F_{(2)1}, \cdots, F_{(m)1}]}{a^j} - V_{j-1}(u_1, u_2, \cdots, u_m).
$$

We shall first show that $V_j(u_1, u_2, \cdots, u_m)$ is of the form $((u^{j+1}))$. This is obviously true for $j = 0$, and we shall assume it proved for $j - 1 = 1, 2, \cdots, h$. Let us take a term in U of degree h in the u's, and let the coefficient of U in V_h be b. The terms obtained from U by replacing u_i by $F_{(i)1}$, $i = 1, 2, \cdots, m$ in V_h, will, with one exception, be of degree higher than h. There will be a single term obtained from U of degree h of the form $b(a^hU)$. Hence in $V_h[F_{(1)1}, \cdots, F_{(m)1}]/a^{h+1} - V_h(u_1, \cdots, u_m)$, the term in U will disappear, and since the same applies to every term of order h, we see that for $j - 1 = h + 1$, the above theorem as to the order of V_j is true. Hence every term of order h in the u's, which appears in the formula (9) for $F_{(i)n}$, will be obtained from V's of subscript less than h. The coefficient of any term of order h in (9) is therefore a polynomial in a^n and is rational except for a^n in the coefficients of $F_{(1)1}, F_{(2)1}, \cdots, F_{(m)1}$. For n any positive integer (9) reduces to an identity and so must coincide with the series for $F_{(i)n}$ given by the method of undetermined coefficients. Since n enters in (9) only in the form a^n, the polynomial in a^n which is in (9) the coefficient of a term in U will coincide for an infinite number of distinct values of a^n, viz., $a^0, a^1, a^2, \cdots, a^j, \cdots$, with the corresponding polynomial obtained by the method of undetermined coefficients. These polynomials must therefore coincide, and the expressions obtained by the two different methods must be identical.

The Solution of the Difference Equation in Terms of the Iteration Series.

4. Having obtained a form for F_n, we may now secure a set of solutions $f_{(i)}$ of (3). For $|a| < 1$, we shall have a definite set of series as the limit of $a^{-n}F_n$, as n becomes positively infinite, while for $|a| > 1$, the same formal series are obtained as the limit of a^nF_n for $n = +\infty$. The series in each case are

$$
(10)\qquad \begin{aligned} G_{(i)}(u_1, u_2, \cdots, u_m) \equiv \Bigg[V_0 - \frac{1}{a - 1} V_1 + \frac{1 \cdot a}{(a - 1)(a^2 - 1)} V_2 \\ - \frac{1 \cdot a \cdot a^2}{(a - 1)(a^2 - 1)(a^3 - 1)} V_3 + \cdots \Bigg], \end{aligned}
$$

as is seen immediately. The G's satisfy the functional equations

$$aG_{(i)}(u_1, \cdots, u_m) = G_{(i)}[F_{(1)1}, F_{(2)1}, \cdots, F_{(m)1}], \tag{11}$$

since, for example,

$$\lim_{n=+\infty} a(a^{-n}F_n) = \lim_{n=+\infty} a^{-n}F_n(F)$$

and similarly for limit, $n = -\infty$. Hence we have,

$$1 + \log_a G_{(i)}(u) = \log_a G_{(i)}[F_1]$$

or, replacing $\log_a G_{(i)}(u)$ by a new notation, $g_{(i)}(u)$,

$$1 + g_{(i)}(u) = g_{(i)}[F_1]. \tag{12}$$

We shall be obliged to restrict the set $g_{(i)}(u)$ to the extent that we shall suppose the system

$$\begin{aligned} x_1 &= g_{(1)}(u_1, u_2, \cdots, u_m), \\ x_2 &= g_{(2)}(u_1, u_2, \cdots, u_m), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_m &= g_{(m)}(u_1, u_2, \cdots, u_m), \end{aligned} \tag{13}$$

to possess an inverse, which we shall denote by

$$\begin{aligned} u_1 &= g_{(1)}^{-1}(x_1, x_2, \cdots, x_m), \\ u_2 &= g_{(2)}^{-1}(x_1, x_2, \cdots, x_m), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_m &= g_{(m)}^{-1}(x_1, x_2, \cdots, x_m). \end{aligned} \tag{14}$$

Combining (12), (13), and (14), we have

$$\begin{aligned} &g_{(i)}^{-1}[1 + x_1, 1 + x_2, \cdots, 1 + x_m] \\ &\qquad = F_{(1)1}[g_{(1)}^{-1}(x_1, \cdots, x_m), \cdots, g_{(m)}^{-1}(x_1, \cdots, x_m)]. \end{aligned} \tag{15}$$

Hence, the set $g_{(i)}^{-1}$, $i = 1, 2, \cdots, m$, constitute a solution of (3).

Consideration of Some Limiting Cases.

5. When a becomes equal to unity, the expression (9) assumes an indeterminate form, which may, however, be readily evaluated. A form for $F_{(i)n}$ when $a = 1$, may also be obtained by a direct method analogous to that used above. We shall have, for $a = 1$,

$$\begin{aligned} F_{(i)n} = \Big\{ F_{(i)0} &+ \frac{n}{1}(F_{(i)1} - F_{(i)0}) \\ &+ \frac{n(n-1)}{1 \cdot 2}(F_{(i)2} - 2F_{(i)1} + F_{(i)0}) + \cdots \Big\} \end{aligned} \tag{16}$$

or as it may be written

$$(17)\quad F_{(i)n} = \left[V_0 + \binom{n}{1}V_1 + \binom{n}{2}V_2 + \cdots + \binom{n}{j}V_j + \cdots\right],$$

where $V_0 = F_{(i)0}$ *and*

$$V_j = F_{(i)j} - \binom{j}{1}F_{(i)j-1} + \binom{j}{2}F_{(i)j-2} - \cdots \pm F_{(i)0}.$$

We shall not consider the case in which $a = 0$. If $|a| = 1$, but $a^j \neq 1$, $j = 1, 2, \cdots$, our original formulæ hold. If, however, $|a| = 1$, and also $a^j = 1$, for some positive integer j, a new case occurs. Let h be the smallest positive integer (different from zero), for which $a^h = 1$. Then the first h terms of (8) and (9) present no difficulty. The $(h + 1)$st term contains however in the denominator the factor $(a^h - 1)$, while this factor will also occur in the numerator of this term, for n any positive integer. If we break off (8) after the hth term the formula will continue to be an identity for $n = 0, 1, 2, \cdots, h - 1$. If $F_{(i)h} = F_{(i)0}$, in which case we always have $a^h = 1$, then the first h terms of (8) will continue to represent $F_{(i)n}$ for every integer value of n, and will therefore, by the general argument, represent a possible definition of $F_{(i)n}$ for all values of n. If $F_{(i)h} \neq F_{(i)0}$, we cannot define $F_{(i)n}$ for all values of n as a series of the type we are considering.*

Discussion of the Convergence of the Series.

6. We shall now demonstrate that (8) is convergent when $F_{(i)1}$ is, $i = 1, 2, \cdots, m$. Let us suppose first that $|a| > 1$. Let us suppose $F_{(i)1}$ dominated by a function $\tilde{F}_{(i)1}$ all of whose coefficients are real and positive. From the method of obtaining the coefficients of $F_{(i)2}$ from those of $F_{(1)1}, F_{(2)1}, F_{(3)1}, \cdots, F_{(m)1}$, we see that the coefficients of $\tilde{F}_{(i)2}$ will be real and positive and will dominate those of $F_{(i)2}$. Similarly for n a positive integer $F_{(i)n}$ will be dominated by $\tilde{F}_{(i)n}$. Now, if $\tilde{F}_{(i)n}$ be determined by (8) as a convergent series for all values of n within a certain region, we see that since $\tilde{a}^n$ is an increasing function of n for $\tilde{a} > 1$, $\tilde{F}_{(i)n}$ will be an increasing function of n for any given values of the independent variables, and in fact each term of (9) is an increasing function of n. We must show that for any set of functions, $\tilde{F}_{(1)1}, \tilde{F}_{(2)1}, \cdots, \tilde{F}_{(m)1}$, whose expansions in power series contain no term with negative coefficients, the $\tilde{V}$'s, obtained as in (9), will have no negative coefficients. Let us suppose, for induction, that $\tilde{V}_{j-1}$ has no negative terms, then we may write

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* Cf. the detailed discussion in the case of one variable, by the author, *Annals of Math.* vol. 17 (1915), p. 35 ff.

$$\tilde{V}_j(u_1, u_2, \cdots, u_m) = \frac{1}{\tilde{a}^j}\{V_{j-1}(\tilde{a}u_1, \cdots, \tilde{a}u_m) + \varphi\} - V_{j-1}(u_1, u_2, \cdots, u_m),$$

where φ consists of partial derivatives of $V_{j-1}(\tilde{a}u_1, \cdots, \tilde{a}u_m)$ multiplied by products of $[F_{(i)1}(u_1, u_2, \cdots, u_m) - \tilde{a}u_i]$ for different or the same values of i, by simply expanding $V_{j-1}(F_{(1)1}, F_{(2)1}, \cdots, F_{(m)1})$ by Taylor's formula for the case of m variables. Now every term of φ has a non-negative coefficient, and since $\tilde{a} > 1$, and $V_{j-1}(u_1, u_2, \cdots, u_m)$ contains, for terms of lowest degree, those of degree j, we see that $(1/a^j)V_{j-1}(\tilde{a}u_1, \tilde{a}u_2, \cdots, \tilde{a}u_m) - V_{j-1}(u_1, u_2, \cdots, u_m)$ contains no terms with negative coefficients. The coefficient in (9) of V_j is itself positive, since, as we have remarked, a^n is an increasing function of n. It therefore suffices to establish the convergence of (9) for a set of dominant functions.

Putting $\tilde{F}_{(i)1} = \tilde{a}u_i/(1 - \tilde{b}s)$, where $s = u_1 + u_2 + \cdots + u_m$, we have

$$\tilde{V}_j = \frac{\tilde{b}^j \cdot 1 \cdot (1 + \tilde{a})(1 + \tilde{a} + \tilde{a}^2) \cdots (1 + \tilde{a} + \cdots + \tilde{a}^{j-1})s^j u_i}{[1 - \tilde{b}s][1 - \tilde{b}(1 + \tilde{a})s][1 - \tilde{b}(1 + \tilde{a} + \tilde{a}^2)s] \cdots [1 - \tilde{b}(1 + \tilde{a} + \cdots + \tilde{a}^{j-1})s]}. \tag{18}$$

But the series $\tilde{F}_{(i)n}$ may be summed in this case in finite terms. We have, by inspection, a possible $\tilde{F}_{(i)n}$ in

$$\tilde{F}_{(i)n} = \frac{\tilde{a}^n u_i}{1 - \tilde{b}\dfrac{1 - \tilde{a}^n}{1 - \tilde{a}}s}, \tag{19}$$

which must coincide with the sum of the series given by (9). Since (18) converges when written as a power series, for $\tilde{b}(\tilde{a}^n - 1)|s| < (\tilde{a} - 1)$, the set of series $F_{(i)1}$, dominated by $\tilde{a}u_i/(1 - \tilde{b}s)$, $\tilde{a} > 1$, will converge at least within this region. For the case $|a| < 1$, we need merely invert the series to obtain $a' = 1/a$ such that $|a'| > 1$, while a' takes the place of a in the inverted series. For $a = 1$, the expression (18) continues to hold while (19) becomes

$$F_{(i)n} = \frac{u_i}{1 - n\tilde{b}s},$$

which converges for $|n|\tilde{b}|s| < 1$. The convergence for $|a| = 1$, but $a^h \neq 1$, $h = 1, 2, 3, \cdots$, depends upon other conditions as well as the convergence of $F_{(i)1}$ as is seen in the case $m = 1$, discussed in the previous paper.

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